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THE EVALUATION IDENTIFICATION IN FUNCTION SPACES

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A study is made of the natural function which maps each point x of a space X to the evaluation function $e_x : Y^X \to Y$ defined by $e_x(f) = f(x)$. A consequence of the results is that βX and υX can both be considered as subspaces of spaces of continuous functions from appropriate domain spaces into I or R, respectively.

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evaluation functions	Σ-product	

One of the standard ways of defining the Stone–Čech compactification, βX, of a space X, is to embed X in a cube and take the closure. A cube is the product of copies of the closed unit interval I, so that the elements of a cube can be considered as functions from a certain index set into I. These functions are not necessarily continuous, even though the index set may have a natural topology.

In this paper we study certain natural functions in the theory of spaces of continuous functions, and obtain as a corollary of our investigation that βX and the real compactification υX can be considered as subspaces of spaces of continuous functions from appropriate domain spaces into I or R, respectively.

Throughout the paper X and Y will denote Tychonoff spaces, Y^X will be the set of all functions from X into Y, and $C(X, Y)$ will be used for the set of all continuous functions from X into Y. If $A \subseteq X$ and $V \subseteq Y$, then we will use the notation $[A, V]$ to mean either

$$[A, V] = \{f \in C(X, Y) \mid f(A) \subseteq V\}$$

or

$$[A, V] = \{f \in Y^X \mid f(A) \subseteq V\}$$

depending on whether we are working with $C(X, Y)$ or Y^X.

1. Set-open topologies

We will say that α is an *X-family* if α is a nonempty family of nonempty closed subsets of X. In addition, we will call α *compact* if the members of α are compact subsets of X, and we will say that α is *proper* if every singleton subset of X is a member of α. Finally, α will be called *regular* if for every $x \in X$ and every neighborhood U of x in X, there exists an $A \in \alpha$ which is a neighborhood of x and which is contained in U.

We will be considering "set-open topologies" on $C(X, Y)$ and Y^X, which can be defined as follows. If α is an X-family then the space $C_\alpha(X, Y)$ is the set $C(X, Y)$ with the topology generated by the subbase

$$\{[A, V] \mid A \in \alpha \text{ and } V \text{ is open in } Y\}.$$

We define the space Y_α^X analogously as the set Y^X with the topology generated by the same subbase, except of course the $[A, V]$ are considered as subsets of Y^X. Note that $C_\alpha(X, Y)$ can be considered as a subspace of Y_α^X.

There are three special kinds of X-families which we will distinguish with names. The family π will always be the family of all singleton subsets of X; the family $\varkappa$ will be the family of all compact subsets of X; and finally the family ω will denote the family of all closed subsets of X. Notice that π is a compact proper X-family, that ω is a proper regular X-family, and that $\varkappa$ is a compact proper X-family which is regular whenever X is locally compact.

For any X-family α, the space Y may be naturally embedded in $C_\alpha(X, Y)$ by identifying each element y of Y with the constant function mapping X onto $\{y\}$. On the other hand, except in special cases, X cannot be embedded in $C_\alpha(X, Y)$ in a natural way. However, we describe next a way that X can be naturally embedded in $C_\beta(C_\alpha(X, Y), Y)$ for certain families α and β.

For each $x \in X$, let us define the *evaluation function at* x to be the function

$$e_x : Y^X \to Y$$

defined by $e_x(f) = f(x)$ for each $f \in Y^X$. When we are working with $C(X, Y)$ we may consider e_x as the function

$$e_x : C(X, Y) \to Y$$

defined by $e_x(f) = f(x)$ for each $f \in C(X, Y)$.

1. Lemma. *If α is a proper X-family, then $e_x : Y_\alpha^X \to Y$ is continuous.*

Proof. Let $f \in Y_\alpha^X$, and let V be a neighborhood of $e_x(f)$. Then $f(x) \in V$, so that $f \in [\{x\}, V]$. If $g \in [\{x\}, V]$, then $e_x(g) = g(x) \in V$, so that $e_x([\{x\}, V]) \subseteq V$. □

It follows from Lemma 1 that $e_x : C_\alpha(X, Y) \to Y$ is also continuous whenever α is proper.

Now we define what we call the *evaluation identification*

$$e: X \to C(C_\alpha(X, Y), Y)$$

by $e(x) = e_x$ for each $x \in X$. It should be clear that if $C(X, Y)$ separates points, then e is one-to-one. This will happen for example when Y contains a nontrivial path. We now determine when e is continuous.

2. Theorem. *If α is a regular X-family and β is a compact $C_\alpha(X, Y)$-family, then $e: X \quad C_\beta(C_\alpha(X, Y), Y)$ is continuous.*

Proof. Let $x \in X$, let $B \in \beta$, and let V be open in Y so that $e(x) \in [B, V]$. Then for every $f \in B$, $f(x) \in V$. Now for each $f \in B$, there exists an $A_f \in \alpha$ such that A_f is a neighborhood of x which is contained in $f^{-1}(V)$. Then $\{[A_f, V] \mid f \in B\}$ is an open cover of the compact set B. Therefore there exist $f_1, \dots, f_n \in B$ such that

$$B \subseteq [A_{f_1}, V] \cup \cdots \cup [A_{f_n}, V].$$

Let $A = A_{f_1} \cap \cdots \cap A_{f_n}$, which is a neighborhood of x. To see that $e(A) \subseteq [B, V]$, let $a \in A$ and $f \in B$. Then there is a k between 1 and n such that $f \in [A_{f_k}, V]$, so that $e(a)(f) = f(a) \in V$. $\square$

Since e is one-to-one when $C(X, Y)$ separates points, we might wonder if e is actually an embedding. By putting one additional assumption on β and requiring that $C(X, Y)$ separate points from closed sets, we do obtain that e is an embedding.

3. Theorem. *If $C(X, Y)$ separates points from closed sets, if α is a regular X-family, and if β is a compact proper $C_\alpha(X, Y)$-family, then $e: X \to C_\beta(C_\alpha(X, Y), Y)$ is an embedding.*

Proof. Let U be open in X, and let $x \in U$. We need to find an open set W in $C_\beta(C_\alpha(X, Y), Y)$ such that $e(x) \in W \cap e(X) \subseteq e(U)$. Since $C(X, Y)$ separates points from closed sets, then there exists an $f \in C(X, Y)$ such that $f(x) \notin \overline{f(X \setminus U)}$. Let $V = Y \setminus \overline{f(X \setminus U)}$, and define $W = [\{f\}, V]$, which is open in $C_\beta(C_\alpha(X, Y), Y)$ since β is proper. Also since $e(x)(f) = f(x) \in V$, then $e(x) \in W$. Now let $z \in X$ with $e(z) \in W$, so that $z \in f^{-1}(V)$. But $V \subseteq Y \setminus f(X \setminus U)$, so that $f(X \setminus U) \subseteq Y \setminus V$. Then

$$X \setminus U \subseteq f^{-1}(f(X \setminus U)) \subseteq f^{-1}(Y \setminus V) = X \setminus f^{-1}(V),$$

so that $f^{-1}(V) \subseteq U$. Therefore $z \in U$, and hence $e(z) \in e(U)$. $\square$

We will consider two special cases where α is a regular X-family. The case that $\alpha = \omega$ will be examined at the end of this section. We consider now the case that $\alpha = \varkappa$ and X is locally compact. In this case the evaluation identification becomes a closed embedding if we assume that Y contains a nontrival path.

4. Theorem. *If X is locally compact, if Y contains a nontrivial path, and if β is a compact proper $C_\kappa(X, Y)$-family, then $e: X \to C_\beta(C_\kappa(X, Y), Y)$ is a closed embedding.*

Proof. First observe that e is an embedding since whenever Y contains a nontrivial path, then $C(X, Y)$ separates points from closed sets. Let $F \in \overline{e(X)}$, and let $\{x_i\}$ be a net in X such that $\{e(x_i)\}$ converges to F in $C_\beta(C_\kappa(X, Y), Y)$. If x is a cluster point of $\{x_i\}$ in X, then $e(x)$ is a cluster point of $\{e(x_i)\}$ in $C_\beta(C_\kappa(X, Y), Y)$, so that $F = e(x)$.

Now suppose, by way of contradiction, that $\{x_i\}$ has no cluster point in X. Then for any compact subset A of X, $\{x_i\}$ is eventually in $X \backslash A$. Let y_0 be a point of Y which is contained in some nontrivial path P in Y, and let f_0 be the constant function mapping X onto $\{y_0\}$. Also let y_1 be an element of P which is distinct from $F(f_0)$, and let V be a neighborhood of $F(f_0)$ which does not contain y_1. Since F is continuous, there exists a basic open set

$$W = [A_1, V_1] \cap \cdots \cap [A_n, V_n]$$

in $C_\kappa(X, Y)$ such that $f_0 \in W$ and $F(W) \subseteq V$. Since X is locally compact, there exists a compact subset A of X such that $A_1 \cup \cdots \cup A_n$ is contained in the interior of A. Then there exists an index i_0 such that for every $i \geq i_0$, $x_i \in X \backslash A$. Since X is completely regular and since y_0 and y_1 are contained in path P, then there exists an $f_1 \in C(X, Y)$ such that $f_1(x) = y_0$ for each $x \in A_1 \cup \cdots \cup A_n$, and $f_1(x) = y_1$ for each $x \in \overline{\{x_i \mid i \geq i_0\}}$. Now $f_1 \in W$, so that $F(f_1) \in V$, and thus $F \in [\{f_1\}, V]$. Then there exists an index i_1 such that for every $i \geq i_1$, $e(x_i) \in [\{f_1\}, V]$, and hence $f_1(x_i) \in V$. But there is an index i with $i \geq i_0$ and $i \geq i_1$, so that $f_1(x_i) = y_1$ while at the same time $f_1(x_i) \in V$; which is a contradiction.

Therefore $\{x_i\}$ has a cluster point in X, so that by our argument in the first paragraph, $F \in e(X)$, and hence $\overline{e(X)} = e(X)$. □

We will be interested in not only having $e: X \to C_\beta(C_\alpha(X, Y), Y)$ be a closed embedding, but also having it be a C-embedding – that is, having the property that all continuous real-valued functions defined on the image of the embedding can be continuously extended to the entire range. From Tietze's extension theorem, we know that whenever e is a closed embedding and $C_\beta(C_\alpha(X, Y), Y)$ is normal, then e is a C-embedding.

We describe now one situation in which $C_\beta(C_\kappa(X, Y), Y)$ is normal. Let X be a locally compact separable metric space, let Y be the real numbers with the usual topology, and let β be a compact proper $C_\kappa(X, Y)$-family. Then $C_\kappa(X, Y)$ is a separable metric space (see [1] and [7]), and therefore $C_\beta(C_\kappa(X, Y), Y)$ is normal (see [6]).

The next theorem will show that it is not really necessary to require $C_\beta(C_\kappa(X, Y), Y)$ to be normal in order to have e be a C-embedding.

If X, Y, and Z are topological spaces where X is embedded in Z, then we will say that X is *C-embedded in Z with respect to Y* if every continuous function from X into Y has a continuous extension from Z into Y.

If $\varphi : X \to Y$ is a function and Z is a topological space, we define the *induced function*

$$\varphi^* : Z^Y \to Z^X$$

by $\varphi^*(g) = g \circ \varphi$ for every $g \in Z^Y$. If φ is continuous, we can consider

$$\varphi^* : C(Y, Z) \to C(X, Z).$$

5. Lemma. *If α is an X-family and β is a Y-family such that $\{\varphi(A) \mid A \in \alpha\} \subseteq \beta$, then $\varphi^* : Z^Y_\beta \to Z^X_\alpha$ is continuous.*

Proof. Let $f \in Z^Y_\beta$, and suppose $\varphi^*(f) \in [A, W]$ in Z^X_α. Since $\varphi(A) \in \beta$, $[\varphi(A), W]$ is open in Z^Y_β. Also $f \in [\varphi(A), W]$ since $f(\varphi(A)) = \varphi^*(f)(A) \subseteq W$. If $g \in [\varphi(A), W]$, then $\varphi^*(g)(A) = g(\varphi(A)) \subseteq W$, so that $\varphi^*([\varphi(A), W]) \subseteq [A, W]$. □

Now we can use the induced function to help us establish our next theorem telling us when e is a C-embedding.

6. Theorem. *If $C(X, Y)$ separates points from closed sets, if α is a regular X-family, and if β is a compact proper $C_\alpha(X, Y)$-family, then $e : X \to Y_\beta^{C_\alpha(X,Y)}$ is a C-embedding with respect to Y.*

Proof. Let $\varphi : X \to Y$ be continuous. We need to find a continuous extension of $\varphi \circ e^{-1} : e(X) \to Y$ to $Y_\beta^{C_\alpha(X,Y)}$. Though $\varphi^* : C_\omega(X, Y) \to C_\alpha(X, Y)$ need not be continuous, we know that $\varphi^{**} : Y_\beta^{C_\alpha(X,Y)} \to Y_\pi^{C_\omega(Y,Y)}$ is continuous by Lemma 5. Now both $e : X \to Y_\beta^{C_\alpha(X,Y)}$ and $e' : Y \to Y_\pi^{C_\omega(Y,Y)}$ are embeddings by Theorem 3.

We first wish to establish that $\varphi^{**} \circ e = e' \circ \varphi$. Let $x \in X$, and let $f \in C_\omega(Y, Y)$. Then

$$\begin{aligned}(\varphi^{**} \circ e)(x)(f) &= \varphi^{**}(e_x(f)) = e_x(\varphi^*(f))\\ &= e_x(f \circ \varphi) = f(\varphi(x))\\ &= e_{\varphi(x)}(f) = e'(\varphi(x))(f) = (e' \circ \varphi)(x)(f).\end{aligned}$$

Therefore $\varphi^{**} \circ e = e' \circ \varphi$ as desired.

It remains to show that $e'(Y)$ is a retract of $Y_\pi^{C_\omega(Y,Y)}$. Let f_0 be the identity function in $C_\omega(Y, Y)$. Consider the evaluation function at f_0:

$$e_{f_0} : Y_\pi^{C_\omega(Y,Y)} \to Y.$$

We know that this is continuous by Lemma 1. Also we see that for each $y \in Y$,

$$(e_{f_0} \circ e')(y) = e_{f_0}(e_y) = e_y(f_0) = f_0(y) = y.$$

This means that

$$e_{f_0} \circ \varphi^{**} : Y_\beta^{C_\alpha(X,Y)} \to Y$$

is our desired continuous extension of $\varphi \circ e^{-1} : e(X) \to Y$ to $Y_\beta^{C_\alpha(X,Y)}$. □

As a consequence we also have the following.

7. Corollary. *Under the hypotheses of Theorem* 6, $e: X \to C_\beta(C_\alpha(X, Y), Y)$ *is a C-embedding with respect to* Y.

We have already considered the case that $\alpha = \varkappa$. In the remainder of this section we consider the case that $\alpha = \omega$.

8. Theorem. *If* β *is a compact* $C_\omega(X, Y)$*-family, then the function* $e: X \to Y_\beta^{C_\omega(X,Y)}$ *has the property that the closure of* $e(X)$ *in* $Y_\beta^{C_\omega(X,Y)}$ *is contained in* $C_\beta(C_\omega(X, Y), Y)$.

Proof. Let F be in the closure of $e(X)$ in $Y_\beta^{C_\omega(X,Y)}$, and let $\{x_i\}$ be a net in X such that $\{e(x_i)\}$ converges to F in $Y_\beta^{C_\omega(X,Y)}$. Then for every $f \in C_\omega(X, Y)$, the net $\{f(x_i)\}$ converges to $F(f)$ in Y.

Now to see that $F \in C(C_\omega(X, Y), Y)$, let $f \in C_\omega(X, Y)$ and let U be a neighborhood of $F(f)$ in Y. Also let V and W be neighborhoods of $F(f)$ in Y so that $\bar{W} \subseteq V$ and $\bar{V} \subseteq U$. Define $A = f^{-1}(\bar{W})$, which is in ω; and thus $[A, V]$ is a neighborhood of f in $C_\omega(X, Y)$. Finally, to see that $F([A, V]) \subseteq U$, let $g \in [A, V]$. Now the nets $\{f(x_i)\}$ and $\{g(x_i)\}$ converge to $F(f)$ and $F(g)$, respectively, in Y. Therefore $\{f(x_i)\}$ is eventually in W, so that $\{x_i\}$ is eventually in $f^{-1}(W) \subseteq A$. But then $\{g(x_i)\}$ is eventually in $g(A) \subseteq V$, which means that $F(g) \in \bar{V} \subseteq U$. $\square$

In our next two corollaries of this theorem, we use the terminology $C(X)$ to mean all real-valued continuous functions on X, and $C^*(X)$ to mean all continuous functions from X into I. For properties of the Stone–Čech compactification βX of X and the realcompactification υX of X, see [3].

9. Corollary. *The embedding* $e: X \to C_\pi^*(C_\omega^*(X))$ *has the property that the closure of* $e(X)$ *in* $C_\pi^*(C_\omega^*(X))$ *is* βX.

Proof. By Theorem 8, the closure of $e(X)$ in $C_\pi^*(C_\omega^*(X))$ is the same as the closure of $e(X)$ in $I_\pi^{C_\omega^*(X)}$, which is compact. Therefore $\overline{e(X)}$ is a compactification of X. Now Corollary 7 tells us that $e(X)$ is C*-embedded in $C_\pi^*(C_\omega^*(X))$, so that $\overline{e(X)} = \beta X$. $\square$

10. Corollary. *The embedding* $e: X \to C_\pi(C_\omega(X))$ *has the property that the closure of* $e(X)$ *in* $C_\pi(C_\omega(X))$ *is* υX.

This can be proved in a manner similar to Corollary 9.

2. Uniform topologies

If (Y, μ) is a uniform space, then there is induced on $C(X, Y)$ a natural uniformity $\{\tilde{M} \mid M \in \mu\}$, where

$$\tilde{M} = \{(f, g) \in C(X, Y) \times C(X, Y) \mid (f(x), g(x)) \in M \text{ for all } x \in X\}.$$

Let $C_\mu(X, Y)$ denote the topological space generated by this uniformity (μ will always be a uniformity on Y rather than an X-family). The uniform space notation which we will use will be similar to that in [5].

All the results in the previous section, except Theorem 4, are true when we replace $C_\alpha(X, Y)$ by $C_\mu(X, Y)$. For example, let us show that the evaluation identification $e: X \to C_\beta(C_\mu(X, Y), Y)$ is continuous whenever β is a compact $C_\mu(X, Y)$-family.

Let $x \in X$, and let $e_x \in [B, V]$, where $B \in \beta$. Then for every $f \in B$, $f(x) = e_x(f) \in V$, so that there exists an $M_f \in \mu$ such that $M_f[f(x)] \subseteq V$. Now let $N_f \in \mu$ such that $N_f \circ N_f \subseteq M_f$, and let U_f be a neighborhood of x such that $f(U_f) \subseteq N_f[f(x)]$. Since B is compact, there exist $f_1, \ldots, f_n \in B$ so that

$$B \subseteq \tilde{N}_{f_1}[f_1] \cup \cdots \cup \tilde{N}_{f_n}[f_n].$$

Define $U = U_{f_1} \cap \cdots \cap U_{f_n}$. To see that $e(U) \subseteq [B, V]$, let $z \in U$ and let $g \in B$. Then there exists a k such that $g \in \tilde{N}_{f_k}[f_k]$, so that $(f_k(z), g(z)) \in N_{f_k}$. Also $f_k(z) \in N_{f_k}[f_k(x)]$, and thus $(f_k(x), f_k(z)) \in N_k$. But then $(f_k(x), g(z)) \in N_{f_k} \circ N_{f_k} \subseteq M_{f_k}$, so that $g(z) \in M_{f_k}[f_k(x)] \subseteq V$. Therefore e is continuous.

Since the proofs of Theorems 3 and 6 can be used to prove the analogous theorems using $C_\mu(X, Y)$, then we obtain the following theorem.

11. Theorem. *If (Y, μ) is a uniform space, if $C(X, Y)$ separates points from closed sets, and if β is a compact proper $C_\mu(X, Y)$-family, then $e: X \to Y_\beta^{C_\mu(X,Y)}$ is a C-embedding with respect to Y.*

We also obtain the analog of Theorem 8.

12. Theorem. *If (Y, μ) is a uniform space and if β is a compact $C_\mu(X, Y)$-family, then the function $e: X \to Y_\beta^{C_\mu(X,Y)}$ has the property that the closure of $e(X)$ in $Y_\beta^{C_\mu(X,Y)}$ is contained in $C_\beta(C_\mu(X, Y), Y)$.*

Proof. Let F be in the closure of $e(X)$ in $Y_\beta^{C_\mu(X,Y)}$, and let $\{x_i\}$ be a net in X such that $\{e(x_i)\}$ converges to F in $Y_\beta^{C_\mu(X,Y)}$. Let $f \in C_\mu(X, Y)$ and let U be a neighborhood of $F(f)$ in Y. Then there exists an $M \in \mu$ such that $M[F(f)] \subseteq U$. Now choose symmetric $N \in \mu$ such that $N \circ N \circ N \subseteq M$. To see that $F(\tilde{N}[f]) \subseteq U$, let $g \in \tilde{N}[f]$. Then for each i, $(f(x_i), g(x_i)) \in N$. Since $\{f(x_i)\}$ and $\{g(x_i)\}$ converge to $F(f)$ and $F(g)$, respectively, there exist i_f and i_g so that for every $i \geqslant i_f$, $f(x_i) \in N[F(f)]$, and for every $i \geqslant i_g$, $g(x_i) \in N[F(g)]$. Then let i_0 be such that $i_0 \geqslant i_f$ and $i_0 \geqslant i_g$. If $i \geqslant i_0$, we have $(F(f), f(x_i)) \in N$ and $(F(g), g(x_i)) \in N$. Thus $(F(f), F(g)) \in N \circ N \circ N \subseteq M$, so that $F(g) \in M[F(f)] \subseteq U$. $\square$

In the next two corollaries, μ will be any uniformity compatible with the usual topology.

13. Corollary. *The embedding $e: X \to C_\pi^*(C_\mu^*(X))$ has the property that the closure of $e(X)$ in $C_\pi^*(C_\mu^*(X))$ is βX.*

14. Corollary. *The embedding $e: X \to C_\pi(C_\mu(X))$ has the property that the closure of $e(X)$ in $C_\pi(C_\mu(X))$ is υX.*

There is some similarity between the way in which βX is obtained in Corollary 13 and the way in which it is obtained in many functional analysis texts. In the latter, X is embedded into the dual space of the space of bounded real-valued continuous functions on X with the supremum metric toplogy, and the closure is taken in this dual space which has the weak* topology.

Finally, let us look at an example. Let X be the space of ordinals less than the first uncountable ordinal ω_1 with the order topology. Every $f \in C(X)$ has the property that there exists an $x \in X$ such that for every $y \geq x$, $f(y) = f(x)$; let $F(f) = f(x)$. Now $F \in R_\pi^{C_\mu(X)}$. To see that $F \in \overline{e(X)}$, where $e: X \to R_\pi^{C_\mu(X)}$ is the evaluation identification, let

$$F \in [\{g_1\}, V_1] \cap \cdots \cap [\{g_n\}, V_n].$$

For each i between 1 and n, there exists an $x_i \in X$ such that for every $x \geq x_i$, $g_i(x) = g_i(x_i)$. Define $x_0 = \max\{x_1, \ldots, x_n\}$. Then for each i,

$$e(x_0)(g_i) = e_{x_0}(g_i) = g_i(x_0) = F(g_i) \in V_i.$$

Therefore $e(x_0) \in [\{g_1\}, V_1] \cap \cdots \cap [\{g_n\}, V_n]$, so that $F \in \overline{e(X)}$. We then know by Theorem 12 that $F \in C_\pi(C_\mu(X))$.

It is clear that $F \notin e(X)$ since if $x \in X$, then we may find an $f \in C(X)$ so that $f(x) \neq F(f)$ and hence $e(x)(f) \neq F(f)$. Now βX is known to be just $[0, \omega_1]$. Also, since X is pseudocompact, then $\upsilon X = \beta X$. Therefore F may be identified with ω_1 in υX, and we can conclude that F is the only element of $C_\pi(C_\mu(X)$ which is in the closure of $e(X)$ but not in $e(X)$.

3. An application

One way in which the evaluation identification may be used is in establishing the dual statements of certain implications. For example, suppose that P is a topological property which is closed hereditary and that Q is a topological property such that whenever X has property Q, then $C_\kappa(X)$ has property P. Then using Theorem 4, we may prove that the dual of this is true – that whenever X is locally compact and $C_\kappa(X)$ has property Q, then X has property P. This technique will be illustrated in the proof of the next theorem.

If $\{X_i\}$ is a family of topological spaces, then a *Σ-product* of $\{X_i\}$ is a subspace $\Sigma(a)$ of $\prod X_i$ with the product topology having the form

$$\Sigma(a)=\{\langle x_i\rangle\in\prod X_i \mid x_i=a_i \text{ for all but countably many } i\},$$

where $a=\langle a_i\rangle$ is some fixed element of $\prod X_i$.

Many spaces can be embedded into Σ-products of separable metric spaces. For example, using Bing's metrization theorem, one can show that every metric space can be embedded into a Σ-product of copies of the unit interval I (see [2]).

In fact many non-metric spaces can be embedded (even as closed subspaces) into Σ-products of separable metric spaces. An example of this is the ordinal space $[0, m)$, where m is an ordinal number. By using the diagonal product map of $\{f_\alpha\}$ where $f_\alpha(\beta)=0$ if $\beta\leqslant\alpha$ and $f_\alpha(\beta)=1$ if $\beta>\alpha$, one may embed $[0, m)$ as a closed subspace of the Σ-product of m copies of I.

Our next theorem says, however, that for a large class of function spaces, the only ones which can be embedded as a closed subspace of a Σ-product of separable metric spaces are those which are already separable metric.

15. Theorem. *Let X be locally compact, and let Y be a separable metric space. If $C_\kappa(X, Y)$ can be embedded as a closed subspace of a Σ-product of separable metric spaces, then $C_\kappa(X, Y)$ is separable metrizable.*

Proof. Gul'ko has shown in [4] that if X is a closed subspace of a Σ-product of separable metric spaces and if Y is a separable metric space, then $C_\kappa(X, Y)$ is Lindelöf. Therefore, under our hypotheses, $C_\kappa(C_\kappa(X, Y), Y)$ is Lindelöf. Then Theorem 4 tells us that the evaluation identification $e: X\to C_\kappa(C_\kappa(X, Y), Y)$ is a closed embedding. Since the Lindelöf property is closed hereditary, then X must be Lindelöf. Now a locally compact Lindelöf space is hemicompact, so that $C_\kappa(X, Y)$ is metrizable (see [1]). Finally, the fact that $C_\kappa(X, Y)$ is separable follows from the fact that it is metrizable and the fact that it is embedded as a closed subspace of some Σ-product of separable metric spaces (see [2]). □

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